Greedy Algorithms

Objective: This module focuses on yet another paradigm, namely, greedy algorithms.

Greedy algorithms solve optimization problems by making the best choice (local optimum) at each step. We shall look at the knapsack problem in various perspectives and we solve them using greedy technique. Note that a greedy algorithm do not always yield optimal solutions, but a feasible solution. For example, if the given optimization problem is a minimization problem, then the solution obtained by greedy is always greater than or equal to the optimal solution and if it is a maximization problem, then the solution obtained by greedy is always less than or equal to the optimal solution. This calls for a proof of correctness if greedy indeed works. In general,

Heuristics + Proof of Correctness = Greedy Algorithms

1 0/1 Knapsack

Given a set of \( n \) objects, say \( x_1, x_2, \ldots, x_n \), along with its weight \( w_1, w_2, \ldots, w_n \) and profits \( p_1, p_2, \ldots, p_n \). The objective is to find \( S \subseteq \{ x_1, x_2, \ldots, x_n \} \) such that \( \text{Profit}(S) = \sum_{i \in S} p_i \) is maximum subject to the constraint \( \sum_{i \in S} w_i \leq W \), where \( W \) is the capacity of the knapsack. This problem is called as 0/1 knapsack because for all \( x_i \), either \( x_i = 1 \) (i.e., the object \( x_i \) is included in the knapsack) or \( x_i = 0 \) (i.e., the object \( x_i \) is not included in the knapsack). Recall that this problem can be solved using the dynamic programming paradigm in pseudo-polynomial time (\( O(nW) \)). Note that if \( W = 2^n \), then the algorithm runs in exponential in \( n \). We shall now try a greedy approach for the given problem and typically greedy algorithms run in polynomial time.

Greedy with respect to weight:

Step 1: Sort the weights in increasing order, say \( w_1 \leq w_2 \leq \ldots \leq w_n \). This step takes \( O(n \log n) \) time.

Step 2: Pick the minimum weight (local optimum), which is \( w_1 \) in this case. Check \( w_1 \leq W \). If so, add \( x_1 \) to \( S \).

Step 3: Check \( w_1 + w_2 \leq W \). If so, add \( x_1 \) and \( x_2 \) to \( S \).

Step 4: Proceed this process until we get the least \( j \) such that \( w_1 + w_2 + \ldots + w_j > W \). Now return \( S = \{ x_1, \ldots, x_{j-1} \} \).

The Steps 2-4 take \( O(n) \). Overall, the above approach takes \( O(n \log n) \) time.

The solution obtained by this greedy approach is a feasible solution, since, \( \sum_{i \in S} w_i \leq W \).

Query: Is the solution obtained by greedy w.r.t weight is optimum ?

Solution: No. Here is a counter example:

The sorted weights and the corresponding profits are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Profits</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

\( w_1 + w_2 + w_3 = 1 + 2 + 4 = 7 > W \) and \( w_1 + w_2 = 1 + 2 = 3 \leq W \) and the corresponding profit is \( 2 + 2 = 4 \).
However, the optimum solution is \( \{x_4\} \) and the profit is 5. Thus, the greedy with respect to weight does not yield optimum always.

The greedy with respect to weight fails because in some cases the objects chosen by the algorithm gives less profit than any optimum solution. So, let us try another greedy approach which is greedy with respect to profit.

**Greedy with respect to profit:**

**Step 1:** Sort the weights in decreasing order, say \( p_1 \geq p_2 \geq \ldots \geq p_n \). Let \( w_1, w_2, \ldots, w_n \) be the corresponding weights for \( p_1 \geq p_2 \geq \ldots \geq p_n \).

**Step 2:** Pick the maximum profit (local optimum), which is \( p_1 \) in this case. Check \( w_1 \leq W \). If so, add \( x_1 \) to \( S \).

**Step 3:** Check \( w_1 + w_2 \leq W \). If so, add \( x_1 \) and \( x_2 \) to \( S \).

**Step 4:** Proceed this process until we get the least \( j \) such that \( w_1 + w_2 + \ldots + w_j > W \). Now return \( S = \{x_1, \ldots, x_{j-1}\} \).

The solution obtained by this greedy approach is a feasible solution, since, \( \sum_{x_i \in S} w_i \leq W \).

**Query:** Is the solution obtained by greedy w.r.t profit is optimum ?

**Solution:** No. Here is a counter example:

The sorted profits and the corresponding weights are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profits</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Weights</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 5, our algorithm outputs \( S = \{x_1\} \) (Since, \( w_1 + w_2 = 5 + 1 = 6 > W \) and \( w_1 = 5 \leq W \) and the corresponding profit is 4. The optimum solution is \( \{x_2, x_3\} \) and the profit is 5. Thus, greedy with respect to profit does not give optimum solution always.

Both the greedy with respect to weight and the greedy with respect to profit fail, because in some cases the objects chosen by the algorithm gives less profit than any optimum. So, now let us try yet another greedy approach with respect to profit per unit weight (profit/weight).

**Greedy with respect to profit/weight:**

**Step 1:** Sort the weights in decreasing order, say \( p_1/w_1 \geq p_2/w_2 \geq \ldots \geq p_n/w_n \).

**Step 2:** Pick the maximum \( p_i/w_i \) (local optimum), which is \( p_1/w_1 \) in this case. Check the corresponding \( w_1 \leq W \). If so, add \( x_1 \) to \( S \).

**Step 3:** Check \( w_1 + w_2 \leq W \). If so, add \( x_1 \) and \( x_2 \) to \( S \).

**Step 4:** Proceed this process until we get the least \( j \) such that \( w_1 + w_2 + \ldots + w_j > W \). Now return \( S = \{x_1, \ldots, x_{j-1}\} \).

The solution obtained by this greedy approach is a feasible solution, since, \( \sum_{x_i \in S} w_i \leq W \).

**Query:** Is the solution obtained by greedy w.r.t profit/weight is optimum ?

**Solution:** No. Here is a counter example:

The sorted weights and the corresponding profits are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profits</td>
<td>7</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>Weights</td>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Profit/weight</td>
<td>2.33</td>
<td>1.8</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 5. Our algorithm picks the first object as an output (Since, \( w_1 + w_2 = 8 \leq W \)) and the corresponding profit is 7. The optimum solution is \( \{x_2, x_3\} \) and the profit is 12. Thus, greedy with respect to profit per unit weight (profit/weight) also does not always give optimum solution.
3 + 5 = 8 > W and \( w_1 = 3 \leq W \) and the corresponding profit is 7. The optimum solution is \( \{x_2\} \) and the profit is 5. Thus, the above greedy with respect to profit/weight does not give optimum always.

**Fine tuning profit/weight:**

Now, let us try to change our strategy slightly by picking the objects as follows. Instead of terminating the algorithm when \( w_1 + w_2 + \ldots + w_j > W \) for least \( j \), neglect \( w_j \) and proceed with \( w_{j+1} \), and check whether the constraint is satisfied or not, terminate this process once all the objects are scanned.

**Query:** Is the solution obtained by greedy approach 2 w.r.t profit/weight is optimum?

**Solution:** No. Here is a counter example:

The sorted profit/weight and the corresponding weights and profits are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profits</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Weights</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Profit/weight</td>
<td>2.33</td>
<td>1.8</td>
<td>1.5</td>
<td>1</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 5. Our algorithm picks the object \( x_1 \) and \( x_4 \) as an output (Since, \( w_1 + w_2 = 3 + 5 = 8 > W \), neglect \( w_2 \) and proceed from \( w_3 \), \( w_1 + w_3 = 3 + 4 = 7 > W \), neglect \( w_3 \) and proceed from \( w_4 \), \( w_1 + w_4 = 3 + 1 = 4 < W \) and the corresponding profit is 8. The optimum solution is \( \{x_2\} \) and the profit is 9. Thus, greedy with respect to profit/weight is not optimum.

## 2 Variant Knapsack

Given a set of \( n \) objects, say \( x_1, x_2, \ldots, x_n \), along with its weight \( w_1, w_2, \ldots, w_n \) and profits \( p_1, p_2, \ldots, p_n \). The objective is to find \( S \subseteq \{x_1, x_2, \ldots, x_n\} \) such that \( |S| \) is maximum subject to the constraint \( \sum_{x_i \in S} w_i \leq W \), where \( W \) is the capacity of the knapsack.

**Greedy with respect to weight**

**Step 1:** Sort the weights in increasing order, say \( w_1 \leq w_2 \leq \ldots \leq w_n \).

**Step 2:** Pick the minimum weight (local optimum), which is \( w_1 \) in this case. Check \( w_1 \leq W \). If so, add \( x_1 \) to \( S \).

**Step 3:** Check \( w_1 + w_2 \leq W \). If so, add \( x_1 \) and \( x_2 \) to \( S \).

**Step 4:** Proceed this process until we get the least \( j \) such that \( w_1 + w_2 + \ldots + w_j > W \). Now return \( S = \{x_1, x_2, \ldots, x_{j-1}\} \).

The solution obtained by this greedy approach is a feasible solution, since, \( \sum_{x_i \in S} w_i \leq W \).

**Example:** The sorted weights and the corresponding profits are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Profits</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 10. Our algorithm picks the objects \( \{x_1, x_2, x_3\} \) as an output (Since, \( w_1 + w_2 + w_3 + w_4 = 1 + 2 + 4 + 5 = 12 > W \) and \( w_1 + w_2 + w_3 = 1 + 2 + 4 = 7 < W \) and \( |S| = 3 \). Optimum Solutions: \( \{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\} \) and \( \{x_1, x_2, x_3\} \). Thus, any optimum solution will pick exactly three objects from \( \{x_1, x_2, x_3, x_4, x_5\} \). Interestingly, this greedy strategy gives optimum always which we shall prove next.

**Claim:** Greedy variant knapsack indeed returns an optimum solution (OPT).

**Proof:** Let \( x_1, x_2, \ldots, x_n \) be the set of objects with weights \( w_1, w_2, \ldots, w_n \) and let the weight of the knapsack be \( W \). Sort \( w_1, \ldots, w_n \) and rename the corresponding objects as \( a_1, \ldots, a_n \). Thus, \( a_1 \leq a_2 \leq \ldots \leq a_n \).
Let \( A = \begin{pmatrix} a_1 & a_2 & \ldots & a_{k-1} & a_k & \ldots & a_n \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix} \) be the output of the algorithm. i.e., \( S = \{a_1, \ldots, a_{k-1} \} \) is the solution obtained by the algorithm.

Let \( B = \begin{pmatrix} b_1 & b_2 & \ldots & b_k & b_{k+1} & \ldots & b_n \\ 1 & 1 & \ldots & 0 & 1 & \ldots & 0 \end{pmatrix} \) be any feasible solution such that \( b_1 = a_1, b_2 = a_2, \ldots, b_n = a_n \).

**Remark:** We work with the binary vector (solution vector) of OPT and the algorithm to show that the number of 1’s in both are same. It is important to note that the behavior of OPT is unknown to the algorithm, i.e., why did OPT set \( b_k = 0 \) and \( b_{k+1} = 1 \), and what is only known to the algorithm is the solution vector output by OPT.

We shall show next that \( \sum b_i = \sum a_i \). Our proof consists of two parts. In sub claim-1, we show that \( \sum b_i \geq \sum a_i \). Further, in sub claim-2, we present a stronger claim: for any feasible solution \( B \) (which includes OPT as well) , \( \sum a_i \geq \sum b_i \).

**Observation 1:** There exists a \( k \in \{1, \ldots, n\} \) such that \( a_i = 1, 1 \leq i \leq k-1 \) and \( a_j = 0, k \leq j \leq n \).

Thus, to show that the algorithm is OPT, we need to establish \( \sum a_i \geq \sum b_i \). Towards this end, we prove the following sub claim.

**Sub Claim 1:** If \( B \) is OPT, then \( \sum b_i \geq \sum a_i \).

Since, the algorithm outputs a feasible solution and since, the number of 1’s in OPT \( \geq \) the number of 1’s in any feasible solution, the above claim is true.

**Sub Claim 2:** If \( B \) is any feasible solution, then \( \sum a_i \geq \sum b_i \).

**Proof of sub claim 2:** Proof is by mathematical induction on \( m \), where \( m \) is the number of bits in which \( A \) and \( B \) differ.

**Base Case:** \( m = 0 \). Clearly, \( \sum a_i \geq \sum b_i \).

**Hypothesis:** Assume that the claim is true if they differ in less than \( m \)-places, \( m \geq 1 \).

**Induction Step:** Let \( A \) and \( B \) differ in \( (m + 1) \)-places, \( m \geq 0 \).

**Observation 2:** Since, \( A \) and \( B \) differ in \( (m + 1) \)-places there exists \( j \) such that \( a_j \neq b_j \). Clearly, it can not be the case that such a \( j \in \{k, \ldots, n\} \). If so, then \( B \) is not a feasible solution. This is true because, if \( B \) matches with \( A \) in the first \( (k-1) \) places and differ in \( j \in \{k, \ldots, n\} \), then clearly \( B \) has exceeded the capacity of the knapsack and hence \( B \) is infeasible.

Therefore, there exists \( j, 1 \leq j \leq k-1 \) such that \( a_j = 1 \) and \( b_j = 0 \).

Find the least \( j \in \{1, \ldots, k-1\} \) such that \( a_j = 1 \) and \( b_j = 0 \).

Now, set \( b_j = 1 \) (For example, if \( A = (1,1,1,0,0,\ldots,0) \) and \( B = (1,1,0,0,0,1,\ldots,0) \), then the modified \( B \) is \( C = (1,1,1,0,0,1,\ldots,0) \)). By doing so, the modified solution vector of \( B \), say \( C \), may become infeasible.

**Case 1:** \( C \) is feasible

The number of positions in which \( A \) and \( C \) differs is \( m \). By the hypothesis, \( \sum a_i \geq \sum c_i \). Note that, \( \sum c_i \geq \sum b_i \). Thus, \( \sum a_i \geq \sum b_i \). Hence, the sub claim is true if \( C \) is feasible.

**Case 2:** \( C \) is not feasible

Since \( C \) is not feasible, there exists a \( j \in \{k, \ldots, n\} \) such that \( a_j = 0 \) and \( c_j = 1 \). Now, set \( c_j = 0 \).

This implies, \( A \) and the modified \( C \), say \( D \), differs in \( (m - 1) \)-places and by the induction hypothesis, \( \sum a_i \geq \sum d_i \). Observe that \( \sum d_i = \sum b_i \). Thus, \( \sum a_i \geq \sum b_i \).

From all the above sub claims, we can conclude that \( \sum a_i = \sum b_i \). Hence, the greedy variant knapsack is OPT.
3 Fractional Knapsack

Given a set of \( n \) objects, say \( x_1, x_2, \ldots, x_n \), along with its weight \( w_1, w_2, \ldots, w_n \) and profits \( p_1, p_2, \ldots, p_n \). The objective is to find a subset \( S \subseteq \{ fx_1, fx_2, \ldots, fx_n \} \) such that \( \text{Profit}(S) = \sum_{x_i \in S} f x_i \cdot p_i \) is maximum, subject to the constraint \( \sum_{x_i \in S} w_i \leq W \), where \( f x_i \) is the fraction of \( x_i \) and \( 0 \leq f x_i \leq 1 \) (so far, we are allowed to include \( x_i \) or to exclude \( x_i \), now, we can choose a fraction of \( x_i \) which varies between 0 and 1), and \( W \) is the capacity of the knapsack. Now let us try to establish a greedy approach for the given problem as follows and we will also check whether the solution obtained by the greedy is optimum or not.

Greedy Strategy 1: with respect to weight:

**Step 1:** Sort the weights in increasing order, say \( w_1 \leq w_2 \leq \ldots \leq w_n \).

**Step 2:** Pick the minimum weight (local optimum), which is \( w_1 \) in this case. Check \( w_1 \leq W \). If so, add \( x_1 \) to \( S \).

**Step 3:** Check \( w_1 + w_2 \leq W \). If so, add \( x_1 \) and \( x_2 \) to \( S \).

**Step 4:** Proceed this process until we get the least \( j \) such that \( w_1 + w_2 + \ldots + w_j > W \). With respect to \( w_j \), calculate \( f x_j = \frac{W - \sum_{i=1}^{j-1} w_i}{w_j} \). Now return \( S = \{ x_1, \ldots, x_{j-1}, f x_j \} \).

The solution obtained by this greedy approach is a feasible solution, since, \( \sum_{x_i \in S} (f x_i \cdot w_i) \leq W \).

**Example:** The sorted weights and the corresponding profits are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Profits</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 5. Our algorithm first chooses \( \{ x_1, x_2 \} \) (Since, \( w_1 + w_2 + w_3 = 2 + 3 + 5 = 10 > W \) and \( w_1 + w_2 = 2 + 3 = 5 \leq W \) and then it chooses \( f x_3 \) amount of \( x_3 \), where, \( f x_3 = \frac{5-(1+2)}{5} = \frac{2}{5} \). Thus, the algorithm outputs \( (x_1, x_2, x_3) = (1, 1, \frac{2}{5}) \) and the corresponding profit is \( 1 + 3 + \frac{2}{5} \times 5 = 6 \), which is same as the optimum.

Query: Is the solution obtained by fractional greedy w.r.t weight is optimum?

Solution: No. Here is a counter example:
The sorted weights and the corresponding profits are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Profits</td>
<td>5</td>
<td>8</td>
<td>15</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 5. Our algorithm picks the first two objects as an output (Since, \( w_1 + w_2 + w_3 = 2 + 3 + 5 = 10 > W \) and \( w_1 + w_2 = 2 + 3 = 5 \leq W \) and the corresponding profit is \( 5 + 8 = 13 \). The optimum solution is \( \{ x_3 \} \) and the profit is 15. Thus, greedy with respect to weight is not optimum.

Greedy Strategy 2: with respect to profit:
**Step 1:** Sort profits in decreasing order, say \( p_1 \geq p_2 \geq \ldots \geq p_n \). Let \( w_1, w_2, \ldots, w_n \) be the corresponding weights for \( p_1 \geq p_2 \geq \ldots \geq p_n \).

**Step 2:** Pick the maximum profit (local optimum), which is \( p_1 \) in this case. Check \( w_1 \leq W \). If so, add \( x_1 \) to \( S \).

**Step 3:** Check \( w_1 + w_2 \leq W \). If so, add \( x_1 \) and \( x_2 \) to \( S \).

**Step 4:** Proceed this process until we get the least \( j \) such that \( w_1 + w_2 + \ldots + w_j > W \). With respect to \( w_j \), calculate \( f_{x_j} = \frac{W - \sum_{i=0}^{j-1} w_i}{w_j} \). Now return \( S = \{ x_1, \ldots, x_{j-1}, f_{x_j} \} \).

The solution obtained by this greedy approach is a feasible solution, since \( \sum_{x_i \in S} (f_{x_i}, w_i) \leq W \).

**Example:** The sorted profits and the corresponding weights are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profits</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Weights</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 5. Our algorithm first chooses \( \{ x_1 \} \) (Since, \( w_1 + w_2 = 4 + 2 = 6 > W \) and \( w_1 = 4 \leq W \)) and then it chooses \( f_{x_2} \) amount of \( x_2 \), where, \( f_{x_2} = \frac{5-4}{2} = \frac{1}{2} = 0.5 \). Thus, the algorithm outputs \( \{ x_1, x_2, x_3 \} = (1, 1/2, 0) \) and the corresponding profit is \( 5 + \frac{1}{2} \times 1 = 5.5 \), where as the \( OPT \) chooses \( \{ x_1, x_3 \} \) and the corresponding profit is 6.

**Query:** Is the solution obtained by fractional greedy w.r.t weight is optimum?

**Solution:** No. The above example is a counter example:

**Greedy Strategy 3: with respect to profit/weight:**

**Step 1:** Calculate the profit/weight for all objects and sort it in decreasing order, say \( p_i/w_i \geq p_2/w_2 \geq \ldots \geq p_n/w_n \).

**Step 2:** Pick the maximum profit/weight (local optimum), which is \( p_1/w_1 \) in this case. Check \( w_1 \leq W \). If so, add \( x_1 \) to \( S \).

**Step 3:** Check \( w_1 + w_2 \leq W \). If so, add \( x_1 \) and \( x_2 \) to \( S \).

**Step 4:** Proceed this process until we get the least \( j \) such that \( w_1 + w_2 + \ldots + w_j > W \). With respect to \( w_j \), calculate \( f_{x_j} = \frac{W - \sum_{i=0}^{j-1} w_i}{w_j} \). Now return \( S = \{ x_1, \ldots, x_{j-1}, f_{x_j} \} \).

The solution obtained by this greedy approach is a feasible solution, since \( \sum_{x_i \in S} (f_{x_i}, w_i) \leq W \).

**Example:** The sorted order of profit/weight, and the corresponding weights and profits are as follows

<table>
<thead>
<tr>
<th>Objects</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profits</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Weights</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Profit/weight</td>
<td>2.33</td>
<td>1.8</td>
<td>1.5</td>
<td>1</td>
</tr>
</tbody>
</table>

Given the weight of the knapsack is 5. Our algorithm first picks the object \( x_1 \) (Since, \( w_1 + w_2 = 3+5 = 8 > W \) and \( w_1 < W \)) and then it picks \( f_{x_2} \), where \( f_{x_2} = \frac{5-3}{5} = \frac{2}{5} \). Thus the algorithm outputs \( \{ x_1, x_2, x_3, x_4 \} = (1, 2/5, 0, 0) \) and the corresponding profit is \( 7 + \frac{2}{5} \times 9 = 10.6 \), which is same as optimum solution.
Claim: Greedy fractional knapsack indeed returns an optimum solution (OPT).

Proof: Let \( x_1, x_2, \ldots, x_n \) be the set of objects with weights \( w_1, w_2, \ldots, w_n \) and profits \( p_1, p_2, \ldots, p_n \), respectively. Let the weight of the knapsack be \( W \). Sort \( p_1/w_1, \ldots, p_n/w_n \) and without loss of generality assume that, \( p_1/w_1 \geq \ldots \geq p_n/w_n \). Let \( A = \) be the output of the algorithm and Let \( B \) be the solution output by any OPT algorithm.

Observation 1: (with respect to \( A \)) There exist a \( k \in \{1, \ldots, n\} \) such that, \( a_j = 1, 1 \leq j \leq k - 1 \), \( a_k, 0 \leq a_k \leq 1 \) and \( a_l = 0, k + 1 \leq l \leq n \).

The objective is to show that profit earned by the algorithm is equal to the profit earned by OPT. Compare \( A \) and \( B \). If they are equal, then there is nothing to prove. If \( A \neq B \):

- Find least \( r \) such that \( a_r \neq b_r \), such \( r \) exists in the range \( \{1, \ldots, k\} \), if not, \( B \) is not feasible. For example, \( A = (1, 1, 1, 2/3, 0, \ldots, 0) \) and \( B = (0, 1, 1/4, 1/7, 1, \ldots, 0) \).

Observation 2: \( a_r > b_r \).

- Increase \( b_r \) to \( a_r \) and accordingly decrease the weights among \( (b_{r+1}, \ldots, b_n) \). Let \( C \) be the modified configuration of \( B \).
- Due to increase in the value of \( r^{th} \) bit \( (b_r \) is increased to \( a_r \)\), there will be an increase in profit at \( r^{th} \) bit. Since, the increase at \( r^{th} \) bit is compensated appropriately by decreasing the values at \((r + 1)^{th}\) bit, \((r + 2)^{nd}\) bit, and so on, there will be a decrease in profit at these bits. We shall now calculate the modified profit as follows.

- Profit gained by \( C = \sum c_i \cdot p_i = \sum b_i \cdot p_i + (c_r - b_r) \times w_r \times \frac{p_r}{w_r} - (b_{r+1} - c_{r+1})w_{r+1} \times \frac{p_{r+1}}{w_{r+1}} - (b_{r+2} - c_{r+2})w_{r+2} \times \frac{p_{r+2}}{w_{r+2}} - \ldots - (b_n - c_n)w_n \times \frac{p_n}{w_n} \)

where, \((c_r - b_r)w_r \times \frac{p_r}{w_r} \) is the increase in profit w.r.t the \( r^{th} \) bit and \(-(b_{r+1} - c_{r+1})w_{r+1} \times \frac{p_{r+1}}{w_{r+1}} - (b_{r+2} - c_{r+2})w_{r+2} \times \frac{p_{r+2}}{w_{r+2}} - \ldots - (b_n - c_n)w_n \times \frac{p_n}{w_n} \) denotes the decrease in profit at \((r + 1)^{th}\) bit.

Since \( p_1/w_1 \geq \ldots \geq p_n/w_n, \frac{p_r}{w_r} \geq \frac{p_{r+1}}{w_{r+1}} \geq \frac{p_{r+2}}{w_{r+2}} \ldots \) Using this we shall simplify the above expression as,

\[
\sum c_i \cdot p_i \geq \sum b_i \cdot p_i + (c_r - b_r) \times w_r \times \frac{p_r}{w_r} - (b_{r+1} - c_{r+1})w_{r+1} \times \frac{p_{r+1}}{w_{r+1}} - (b_{r+2} - c_{r+2})w_{r+2} \times \frac{p_{r+2}}{w_{r+2}} - \ldots - (b_n - c_n)w_n \times \frac{p_n}{w_n}
\]

Thus, \( \sum c_i \cdot p_i \geq \sum b_i \cdot p_i + \frac{p_r}{w_r} \left[ (c_r - b_r)w_r - \sum_{i=r+1}^{n} (b_i - c_i) \cdot w_i \right] \)

Note that \( (c_r - b_r)w_r - \sum_{i=r+1}^{n} (b_i - c_i) \cdot w_i \) = 0 as increase at \( r^{th} \) bit is compensated with bits in \((r + 1), \ldots, n\).

Therefore, \( \sum c_i \cdot p_i \geq \sum b_i \cdot p_i \Rightarrow \) we now have either \( \sum c_i \cdot p_i > \sum b_i \cdot p_i \) or \( \sum c_i \cdot p_i = \sum b_i \cdot p_i \) If \( \sum c_i \cdot p_i > \sum b_i \cdot p_i \), then it contradicts the optimality of \( B \). Thus, \( \sum c_i \cdot p_i = \sum b_i \cdot p_i \).

- Now, check whether \( A = C \) or not. If \( A = C \), then the proof is done. If \( A \neq C \), then repeat the above process till \( A = C \). Note that at the end of the first iteration of the above proof, if \( A \neq C \), then they will match at first \( r \) places and may differ at any of the places in the range \( r + 1, \ldots, n \). By repeating the above argument (at max \( n \) iterations), we eventually see that the behavior of the greedy algorithm and optimum are same.

- Thus, we conclude the output of greedy is indeed optimum.
Acknowledgements: Lecture contents presented in this module and subsequent modules are based on the following text books and most importantly, author has greatly learnt from lectures by algorithm exponents affiliated to IIT Madras/IMSc; Prof C. Pandu Rangan, Prof N.S. Narayanaswamy, Prof Venkatesh Raman, and Prof Anurag Mittal. Author sincerely acknowledges all of them. Special thanks to Teaching Assistants Mr. Renjith.P and Ms. Dhanalakshmi.S for their sincere and dedicated effort and making this scribe possible. Author has benefited a lot by teaching this course to senior undergraduate students and junior undergraduate students who have also contributed to this scribe in many ways. Author sincerely thanks all of them.

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