SYNOPSIS OF

STUDY OF NONLINEAR SYSTEMS USING APPROXIMATION METHODS

A THESIS

to be submitted by

SAGAR ZEPHANIA C F (ROLL NO.:PHY18D002)

for the award of the degree

of

DOCTOR OF PHILOSOPHY

INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, DESIGN AND MANUFACTURING, KANCHEEPURAM

NOVEMBER 2021

1 Introduction

Most of the physical systems are nonlinear in nature, and they execute vibrations which is a deviation from the exactly solvable harmonic oscillator (HO) due to the presence of nonlinearity and are termed as anharmonic oscillators(AHO). AHOs are widely used to describe various physical phenomena at classical as well as quantum level [1, 2, 3, 4, 5, 6, 7] such as the molecular spectra [8], hydrogen-bonded solid [9], nuclear shape oscillation [10], quark-model [11] etc. Korteweg-de Vries-Burgers (KdVB) equation [12] is one of the fascinating $(1 + 1)$ dimensional partial differential equations to the researchers because all three effects, such as nonlinearity, dissipation, and dispersion, appear together. KdVB equation describes several physical phenomena such as propagation of undular bores in shallow water[13, 14], the flow of liquid containing gas bubble, waves in an elastic tube filled with viscous fluid, nonlinear plasma wave. KdVB reduces to KdV and Burgers equation when dissipation and dispersion term tends to zero, respectively. Kadmotsev-Petviashvili (KP) equation is the generalized form of the KdV equation with two space variables that explain the general weakly dispersive waves. The broad applicability of such equations has attracted the attention of researchers in science and engineering, even in present days. However, such problems are usually challenging to solve, either numerically or analytically.

Systems with nonlinearity are mostly not exactly solvable because of the presence of nonlinear terms in the governing equations of the system. Although solving the nonlinear problems numerically is sometimes easy, one desires to get the analytical solutions of such problems as they carry more information and give a better insight into the system. Perturbation method [15, 16] is a widely used method for finding an approximate analytical solution to the complex nonlinear systems, especially with the nonlinear term appearing as an additional term of small order to an exactly solvable problem. As the equations for many nonlinear systems do not have a small perturbation term, the application of the perturbation technique is highly restricted. There are many techniques for solving nonlinear oscillator problems, such as the harmonic balance method [17], weighted linearization method [18], modified Lindstedt-Poincare method [19], Adomian decomposition method [20], and so on, to yield approximate analytical solutions to such problems. Most of the methods are somewhat plagued with the complexity of calculation and fail to properly handle problems with strong nonlinearity. Therefore, an approximate method is still looked for, which is easy to calculate but produces highly accurate results. Liao [21] proposed an analytical method in 1992, known as the homotopy analysis method (HAM), which introduces an embedding parameter to construct a homotopy of the given system and then analyzes it using the Taylor formula. HAM provides a way to ensure the convergence of the solution series considering an auxiliary parameter (h) in the construction of homotopy. J. H. He coupled the idea of homotopy with perturbation method (HPM) for solving initial/boundary value problems for non-linear systems [22]. The solution is given in an infinite series in this method, usually converging to an accurate solution. [23]. Applicability of HPM in solving or analyzing nonlinear systems both in the classical [24, 25] as well as the quantum mechanical domain [26] is found to be easy and satisfactory.

2 Motivation

HPM is a simple approximation method to find an accurate analytical solution for anharmonic oscillators. But it is found not to be so efficient for the strongly nonlinear oscillator. Nofal et al. [27] employed FAF followed by EBM, to study some physically relevant anharmonic oscillators with strong anharmonicities and concluded that FAF-EBM method gives better accuracy in comparison to that obtained by using EBM alone. Recently, Aboodh transform-based HPM (AT) [28] has been used to find approximate analytical solutions to various physically relevant anharmonic oscillators and is found to produce results with better accuracy in comparison to those obtained from established approximation methods, namely, EBM [29] and FAB-EBM for all cases. But a slight deviation of AT solutions from the numerical solution is observed for the autonomous conservative oscillator (ACO). This indicates a need for a method that will be simple in performing calculations but yield results with improved accuracy. A large pool of physical problems exists, such as shallow-water waves, nonlinear plasma waves, etc., which involve higher dimensions. A few attempts have been made so far to address such problems in the light of HPM.

3 Scope and Objective

3.1 Scope

Several attempts are made to improve the accuracy of HPM, such as by (i) coupling HPM with some other methods like variational method (ME) [30], Laplace transforms

(LHPM) [31], Aboodh transforms (AT) [28] etc., (ii) considering an expansion of the frequency term [26] and (iii) adopting an additional parameter in HPM [32] to get a better convergence as it is done in HAM. But a high accuracy in the solution is yet to achieve maintaining the simplicity of calculation.

3.2 Objective

The objective of the thesis are as follows:

- To consider an expansion of frequency term ($\Lambda = 1/\omega^2$) to improve the accuracy of calculation and to adopt an auxiliary parameter (h) [21] to control the convergence in the framework of HPM, simultaneously. Laplace transformation may use for the calculation of solving nonlinear differential equations to simplify further. The Laplace transform-based HPM with auxiliary parameter h and Λ-expansion (LH) is expected to solve the governing equation of strongly nonlinear equations with high accuracy but without much computational rigour. It is intended to calculate the displacement and frequency using LH, especially for ACO, for which solution obtained from AT show significant deviation from the numerical solution.
- It is intended to study higher-dimensional problems governed by KdVB and KP equations involving soliton solutions using LH.
- To formulate an improved method (LHh) over LH for getting an analytical solution to the AHO with the generalized polynomial (symmetric and asymmetric) as the restoring force considering an expandable h in the framework of LH.
- To estimate the range of parameters of the resorting forces, which yields a solution using the new method with good accuracy.
- It is intended to study the system's stability from a variation of critical points with the change of each force parameter at a time, which helps one understand the state of stability of the system for a particular value of that parameter. Such a study helps tune a parameter to put a physical system in a desired state of stability.
- To study phase portrait to understand the stability of the systems.

4 Description of the research work

The improved homotopy perturbation method (LH) is proposed considering an expansion of frequency term ($\Lambda = 1/\omega^2$) and a convergence control parameter (h) in the framework of HPM, which is further coupled with the Laplace transform to make the calculation easier. The LH method is applied to get the approximate analytical solution to the ACO. The LH method is also used to get soliton solution for KdVB and KP equation for different cases.

4.1 Autonomous Conservative Oscillator

A mechanical system often consists of slender cantilever beams with a flexible root carrying an intermediate lumped mass and often undergo large-amplitude vibrations. Such vibrating systems are known as an autonomous conservative oscillator which is governed by a strongly nonlinear differential equation with fifth-order nonlinearity,

$$\ddot{x}(1+\epsilon x^2+\alpha x^4)+\lambda x+\epsilon x\dot{x}^2+2\alpha x^3\dot{x}^2+\beta x^3+\gamma x^5=0. \tag{1}$$

Solution to Eq .(1) is obtained employing LH with initial conditions, $x(0) = a$ and $\dot{x}(0) = 0$. The displacement and frequency of ACO with first-order approximation are,

$$\begin{aligned} x_{LH}(t) &= a\cos\omega t+\frac{h}{128}\left[8\epsilon a^3+7\alpha a^5-4\Lambda_0\beta a^3-5\gamma\Lambda_0 a^5\right](\cos\omega t-\cos 3\omega t) \\ &+\frac{ha^5}{384}[3\alpha-\gamma\Lambda_0](\cos\omega t-\cos 5\omega t), \end{aligned} \tag{2}$$

where, $\omega = (\Lambda_0+\Lambda_1)^{-1/2}$, with $\Lambda_0 = \frac{8+3a^4\alpha+4a^2\epsilon}{8\lambda+5a^4\gamma+6a^2\beta}$, $\Lambda_1 = \frac{a^2h}{192(6a^2\beta+5a^4\gamma+8\lambda)}[96a^2\alpha-15a^6\alpha^2+96\epsilon-12a^4\alpha\epsilon-\Lambda_0(48\beta+64a^2\gamma+138a^4\alpha\beta+150a^6\alpha\gamma+144a^2\beta\epsilon+156a^4\gamma\epsilon+96a^2\alpha\lambda+96\epsilon\lambda)+\Lambda_0^2(72a^2\beta^2+174a^4\beta\gamma+105a^6\gamma^2+48\beta\lambda+64a^2\gamma\lambda)]$. The displacement and frequency obtained from LH are compared with the same given by AT, Hamiltonian approach technique (HT)[33], and that obtained by fourth-order Runge-Kutta (RK4) method. The reliability of RK4 solutions is checked by changing the mess size and the accuracy by comparing the same from the Mathematica function NDSolve. It is noticed that LH gives much better accuracy (at least an order of magnitude) than other methods compared with all sets of parameters considered here. Fig.1 presents such comparisons for parameter sets as a sample. The contribution from the second-order approximation in LH is found not significant. LH has been found to be trustworthy

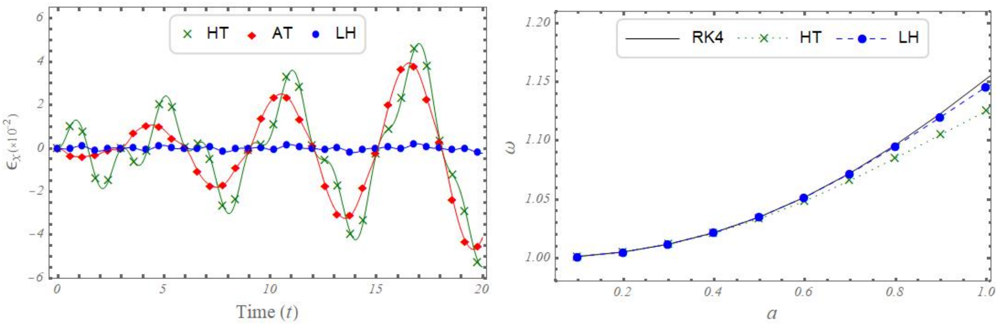

Figure 1: The left panel displays absolute errors ($\epsilon_x = x_{RK4} - x_{approx}$) in the approximate solutions with respect to RK4 for the parameter set $a = 0.2$, other parameters (OP) $= 0.2$. The right panels shows the variations ω with a for OP$= 1$

for analyzing the phase portrait of the system. The results of Duffing oscillator, which is a special case of ACO ($\alpha = \epsilon = \gamma = 0$), are compared with the modified newton method coupled with the harmonic balance (SN) method [34]. As for accuracy, the second-order solution obtained from both SN (SN2) and LH (LH2) are highly accurate, but the LH2 analytical expressions of the solutions and the calculations are very simple in comparison to SN2.

4.2 Korteweg-de Vries-Burgers equation

LH method is applied to find out the analytical approximate soliton solutions for the KdVB equation,

$$u_t + \epsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0. \tag{3}$$

Simple and compact analytical expressions for the solutions are obtained not only for the leading order but also for the higher-order approximations, which mimic the profiles of the exact results. LH results are compared with those obtained from MVIA-II [35] which is compared with several presently available methods and is claimed to be the best among them (improved by order of magnitude). It is noted that for the KdVB equation (when both μ and ν are equally prominent), LH produces solutions using first-order (at most second-order) approximation with a similar or better accuracy (by order of magnitude) in comparison to the fifth-order MVIA-II solutions, as shown in Table 1. For $\nu = 0$ (KdV), the third-order LH gives the accuracy of the solution similar to the fourth-order MVIA-II solution. For the case $\mu = 0$ (Burger's equation), fourth-order LH solutions are found to have at least 45% less error with respect to fourth-order MIVA-II. . Therefore, we may conclude that LH is not only a simple method to find an

Table 1: Comparison of absolute errors ($\delta(approx) = |u_{exact} - u_{approx}|$) of the solution of Eq. (3) with $\epsilon = 1$ and $t = 100$ for different parameters obtained from fifth-order MVIA-II [35], first-order and second-order LH. Here, $y(-n)$ indicates $y \times 10^{-n}$.

$\nu(=\mu)$	x	u_{exact}	δ(MVIA-II)	δ(LH)	δ(LH2)
0.001	0	-0.000360575307696737	9.426 (-08)	9.669 (-08)	9.426 (-08)
	50	-0.000479999999020187	1.262 (-14)	1.261 (-14)	1.261 (-14)
	100	-0.000480000000000000	1.084 (-19)	6.520 (-23)	2.601 (-23)
0.01	0	-0.003656898008681638	7.925 (-06)	1.030 (-05)	7.949 (-06)
	50	-0.004799999991012855	1.227 (-12)	1.222 (-12)	1.226 (-12)
	100	-0.004799999999999999	8.674 (-19)	2.519 (-22)	2.528 (-21)
0.1	0	-0.040986399012315563	3.338 (-05)	1.732 (-03)	5.558 (-05)
	50	-0.047999999962120477	9.821 (-11)	9.267 (-11)	9.314 (-11)
	100	-0.047999999999999999	6.939 (-15)	1.910 (-19)	1.920 (-19)
1	0	-0.409863990123155634	3.338 (-04)	1.720 (-02)	5.558 (-04)
	50	-0.479999999621204775	9.851 (-10)	9.261 (-10)	9.314 (-10)
	100	-0.479999999999999999	1.898 (-14)	1.909 (-18)	1.921 (-18)

approximate analytical solution, but also it yields a highly accurate solution for (1+1)-dimensional problems with solitons solution.

4.3 Kadomtsev-Petiashvili equation

The Kadomtsev-Petviashvili (KP) equation is the (2+1)-dimensional nonlinear partial differential equation and is given by,

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{4}$$

LH method is applied to find the approximate solution of Eq.(4) and found to be nicely matching with the exact solution. Absolute error in the first-order and second-order solution obtained from LH for different values of x, y, and t are calculated. It is seen that the LH yielded a very accurate result for all values of time considered for the study. The absolute error is very small (order of 10^{-5}) for time, $t = 0.01$, and it is increased to order of 10^{-3} when $t = 0.1$. In Fig 2, the absolute errors for the first and second-order LH solutions with exact solution [36] for $x = 0.1$ are displayed. Consideration of second-order improves the accuracy by order of magnitude two.

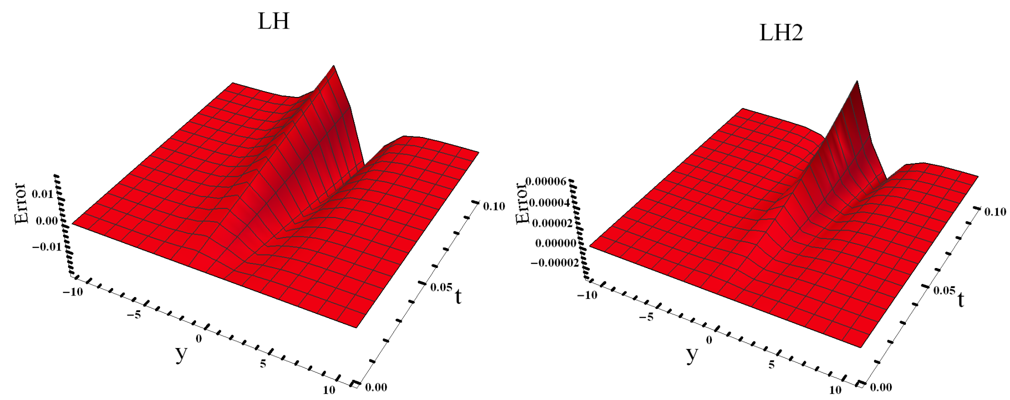

Figure 2: The absolute error ($u_{exact} - u_{LH}$) of the solution obtained from the first (left) and second (right) order LH method for $x = 0.1$.

4.4 Symmetric and asymmetric oscillators with polynomial restoring forces

Anharmonic oscillators with polynomial restoring force,

$$F(x) = -\sum_{i=1}^{m} d_i x^i, \quad m = 1, 2, 3 \cdots, \tag{5}$$

demand a method for solving the corresponding governing equation, which can handle the symmetric and asymmetric systems in a generalized way with high accuracy. LH method is improved by considering the expansion of convergence control parameter h in terms of p. Expansion of the frequency term, ($\lambda = 1/\omega$) instead of ($\Lambda = 1/\omega^2$) is considered to widen the scope of the application of the method to the systems containing a term with odd power of $\dot{x}$.

4.4.1 Asymmetric system ($m = 2$)

Let us consider the lowest order of nonlinearity ($m = 2$) in Eq. (5), which represents an asymmetric AHO, and the corresponding governing equation of the system is,

$$\ddot{x}(t) + b_1 x + b_2 x^2 = 0, \tag{6}$$

where, $b_i = d_i/M$, M is the mass. Variations of root mean square (rms) errors in x obtained from LHh, and second-order LHh (LHh2) with b_2 for $a_R = 1$ and $b_1 = 1$ are presented in the left panel of Fig.3. Here, a_R is the perpendicular distance from the equilibrium position to the right turning point of the potential. The LHh method is found

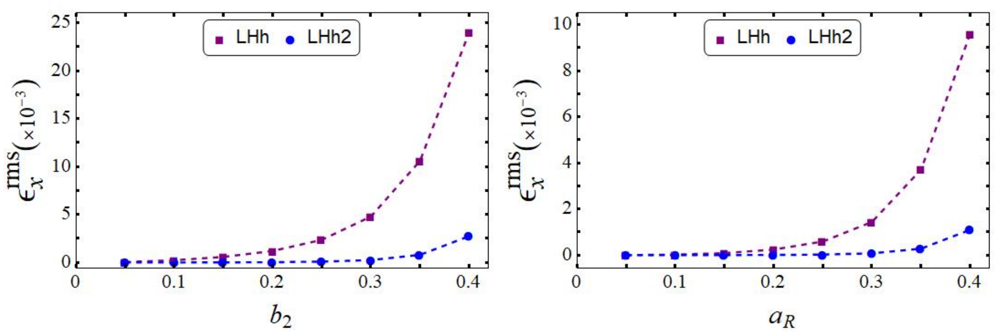

Figure 3: Comparisons of root mean square error (ϵ_x^{rms}) in $x_{LHh}(t)$ and $x_{LHh2}(t)$ with respect to x_{RK4} are displayed for different values of b_2 and a_R.

to yield accurate results, and a significant improvement happens when the second-order terms are considered. A similar trend is obtained for variation of a_R. Frequencies of the AHO (6) are computed using LHh considering first-order approximation (ω_{LHh}) for different values of b_2 and amplitudes. The results are compared and found that ω_{LHh} matches closely with exact frequency (ω_{EX}) [37] (up to fifth decimal) for low values of a for all b_2. As a_R is approaching the maximum value of a_R ($a_{R_{max}}$), ω_{LHh} deviates very much from the corresponding exact values because the system becomes very asymmetric (highly anharmonic) and tends to move away from that potential well where it is confined. The results are seen to be improved by order of magnitude if the second-order approximation is considered.

4.4.2 Symmetric system

With $m = 5$, and $b_2 = b_4 = 0$, the governing equation of the AHO becomes

$$\ddot{x}(t) + b_1 x + b_3 x^3 + b_5 x^5 = 0. \tag{7}$$

LHh solutions are compared with some available other approximate methods considering RK4 results as a benchmark for different parameter sets. For all cases, LHh yields accurate results in comparison to other methods. For an example deviations (ϵ_x) of displacements obtained from LHh and coupled method of HPM and variational approach (ME) with respect to its values calculated using RK4 is displayed in Fig.4 for amplitudes $a = 5$, and $a = 10$. It is clear that for the large amplitudes ($a = 5$ and $a = 10$), ϵ_{LHh2} still remains close to zero whereas the ϵ_{ME} increases to a larger value with the increase in a. The solution obtained from ME gets out of phase with respect to the RK4 solution very fast.

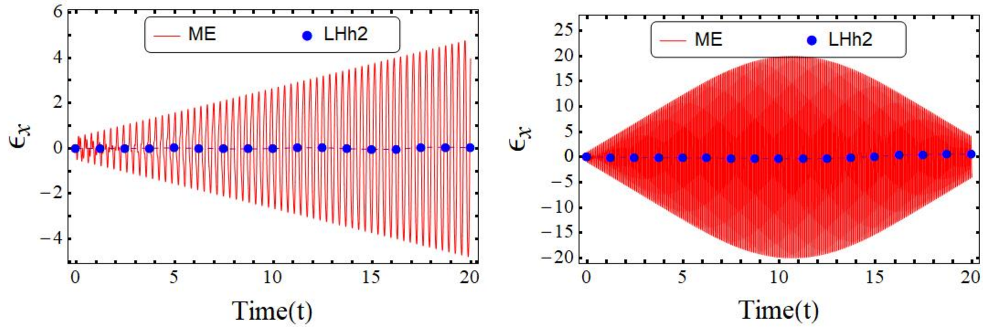

Figure 4: Plot of absolute errors in the approximate solutions with respect to RK4 ($\epsilon_{x_{ME}}$ and $\epsilon_{x_{LHh2}}$) for $a = 5$ (left) and $a = 10$ (right) keeping $b_3 = b_5 = 1$

4.4.3 Asymmetric system with higher-order anharmonically

Let us consider more general polynomial restoring ($m = 5$), and corresponding governing equation is given by,

$$\ddot{x}(t) + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 = 0, \tag{8}$$

which is not exactly solvable. Here, three sets of parameters are considered, which corresponds to three different kinds of potential wells, namely, single well (SW: $b_1 = b_2 = b_3 = b_4 = b_5 = 1$), double well (DW: $b_1 = b_2 = b_4 = b_5 = 1$ and $b_3 = -3$) and triple well (TW:$b_1 = b_4 = b_5 = 1, b_2 = -2.4$ and $b_3 = -4$). Time evolution of displacement x_{LHh} and x_{LHh2} for the parameter set SW compared with that obtained by RK4 and noticed that both curves from the approximate solutions coincide with each other. A small disagreement of x_{LHh} with the corresponding RK4 values are noticed for DW and TW (deviation increases with the increase of the complexity of the structure of the potential). But $\epsilon_{x_{LHh2}}$ remains very close to zero for all three cases throughout the time span considered here.

The system's stability is studied from a variation of critical points with the change of each force parameter at a time. The rms deviation of x due to variation all five parameters in Eq. (8) individually for a wide range of values are studied to quantify the effect of each parameter (the strength of nonlinearity) on the accuracy of the approximate result, which facilitates understanding the behavior of the system. It is also noted that the efficiency of LHh becomes low when the system is in proximity to an unstable point. Therefore, there is a need to improve the LHh further.

Phase portrait obtained from LHh and LHh2 and compared with the corresponding phase portrait from RK4 for the AHO for the three kinds of potential well in Fig.5. It is

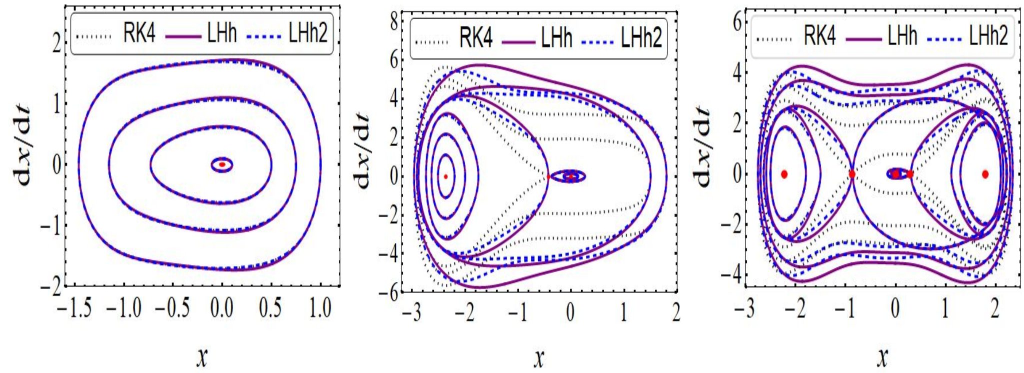

Figure 5: Phase plots for SW (left), DW (middle), and TW (right).

evident from the figure that the phase curves from both LHh and LHh2 match extremely well with the RK4 one for the SW potential. Approximate phase curves deviate from those of RK4 whenever the DW or TW potentials are considered. However, it shows the periodic nature of the system. There are several attempts to improve the accuracy of HPM. To gauge the efficiency of each approach, the frequencies computed for Eq. (8) employing HPM, HPM plus $1/\omega$-expansion (Hω), HPM plus $1/\omega^2$-expansion with h (LH), $H\omega$ with h-expansion (LHh), second-order LH (LH2) and LHh2 for different values of a^* (initial value of displacement) and presented in Table 2. This study reveals that consideration of expansion of the x, frequency term, $\lambda = 1/\omega$ and h gives the best combination. The LHh Method not only improves the improvement of accuracy of the solution but also widens the scope of the applicability of the method (LHh) to problems involving odd powers of $\dot{x}$ where HωS or LH fails.

Table 2: Comparison frequency obtained from HPM and its different modified versions for the Eq.(8) with OP $= 1$ and absolute errors (in parenthesis) are tabulated

a^*	Exact	HPM	Hω	LH	LHh	LH2	LHh2
0.1	0.9995	1.0005 (0.0010)	0.9995 (0.0000)	0.9995 (0.0000)	0.9995 (0.0000)	0.9995 (0.0000)	0.9995 (0.0000)
1	1.6267	1.6689 (0.0419)	1.6276 (0.0009)	1.6281 (0.0014)	1.6265 (0.0008)	1.6264 (0.0003)	1.6268 (0.0001)
10	78.0249	82.5677 (4.5429)	78.4497 (0.4248)	78.4161 (0.3912)	78.3040 (0.2791)	77.9206 (0.10143)	78.0291 (0.0042)
100	7498.54	7539.75 (439.09)	7536.54 (31.25)	7529.80 (37.99)	7525.54 (27.10)	7488.42 (10.11)	7498.97 (0.42)

5 Conclusion and Future Work

5.1 Conclusion

An expansion of frequency term ($\Lambda = 1/\omega^2$) is considered to improve the accuracy of calculation, and an auxiliary parameter (h) is adapted to control the convergence in the framework of HPM simultaneously. Laplace transform is applied for making the calculation of solving nonlinear differential equations further simple. The Laplace transform-based HPM with auxiliary parameter h and Λ-expansion (LH) is applied to find the approximate analytical solution of ACO, which is a highly nonlinear system. It is found that the values of x and ω were calculated using LH (first-order) with an accuracy of at least an order of magnitude better than those of AT and HT for the parameter sets considered. LH is found to be trustworthy for analyzing phase portraits of a system. The contribution of higher-order approximation is found to be negligible in this problem.

KdVB equation is used to model waves with dispersion, dissipation, and nonlinearity in (1+1)-dimension. LH is employed to solve this equation for different special cases and found to yield approximate analytical results with higher accuracy and simpler way in comparisons to those by some current methods, as discussed in 4.2.

The Kadomtsev-Petviashvili (KP) equation is a (2+1)-dimensional nonlinear partial differential equation and mostly describes weakly dispersive waves. LH is found to produce highly accurate and compact solutions for this problem too. The absolute error in LH with respect to the exact solution is very low, such as the maximum error from the first-order calculation is of the order of 10^{-3} (for the parameter range considered)

LH method is improved (LHh) by considering the expansion of convergence control parameter h in terms of p. Expansion of the frequency term, $\lambda = 1/\omega$ instead of $\Lambda = 1/\omega^2$ is considered to widen the scope of the application of the method It is found that the values of x and ω calculated using LHh agree very closely with the corresponding exact (or RK4) results even for both symmetric and asymmetric systems It is noted that phase plots using LHh agree with those obtained from RK4 for the single-well, but some deviations are noticed for double and triple well systems. The enhancement of accuracy in ω for different modified schemes of HPM is studied, and it is found that the performance of LHh is better than other versions of HPM. The rms error with the variation of parameters is studied, which enables us to estimate the range of parameters of the resorting forces where the new method yields good accuracy. The stability of the system is studied from a variation of critical points with the change of each force parameter at a time. This study helps to understand the state of stability of the system for a particular value of that parameter which may help one tune a parameter to put a physical system in a desired state of stability.

5.2 Future scope

1. Apply the improved homotopy perturbation method (LHh) in velocity-dependent oscillators such as the damped Duffing oscillator, and work progresses

2. Apply the LHh to physically relevant problems of other fields of science such as fluid dynamics, quantum mechanics, etc.

REFERENCES

[1] R. Bonham and L. Su, "Use of hellmann feynman and hypervirial theorems to obtain anharmonic vibration rotation expectation values and their application to gas diffraction," *J. Chem. Phys.*, vol. 45, no. 8, p. 2827, 1966.

[2] C. M. Bender and T. T. Wu, "Analytic structure of energy levels in a field-theory model," *Physical Review Letters*, vol. 21, no. 6, p. 406, 1968.

[3] K. Manimegalai, S. Paul, M. Panja, and T. Sil, "Study of the sextic and decatic anharmonic oscillators using an interpolating scale function," *The European Physical Journal Plus*, vol. 135, no. 2, pp. 1–15, 2020.

[4] R. Adhikari, R. Dutt, and Y. Varshni, "Exact solutions for polynomial potentials using supersymmetry inspired factorization method," *Physics Letters A*, vol. 141, no. 1, pp. 1–8, 1989.

[5] B. Ravindra and A. K. Mallik, "Role of nonlinear dissipation in soft duffing oscillators," *Phys. Rev. E*, vol. 49, pp. 4950–4954, Jun 1994.

[6] S. Dixit, A. Sharma, and M. D. Shrimali, "The dynamics of two coupled van der pol oscillators with attractive and repulsive coupling," *Physics Letters A*, vol. 383, no. 32, p. 125930, 2019.

[7] B. Pratiher and S. K. Dwivedy, "Nonlinear Vibration of a Magneto-Elastic Cantilever Beam With Tip Mass," *Journal of Vibration and Acoustics*, vol. 131, 02 2009. 021011.

[8] F. Wall and G. Glockler, "The double minimum problem applied to the ammonia molecules," *The Journal of Chemical Physics*, vol. 5, no. 5, pp. 314–315, 1937.

[9] R. L. Somorjai and D. Hornig, "Double-minimum potentials in hydrogen-bonded solids," *The Journal of Chemical Physics*, vol. 36, no. 8, pp. 1980–1987, 1962.

[10] P. Buganu and R. Budaca, "Shape phase transition in γ-rigid prolate nuclei," in *AIP Conference Proceedings*, vol. 1796, p. 020008, AIP Publishing, 2017.

[11] C. Quigg and J. L. Rosner, "Quantum mechanics with applications to quarkonium," *Phys. Rept.*, vol. 56, no. FERMILAB-PUB-79-022-T, pp. 167–235, 1979.

[12] C. H. Su and C. S. Gardner, "Korteweg-de vries equation and generalizations. iii. derivation of the korteweg-de vries equation and burgers equation," *Journal of Mathematical Physics*, vol. 10, no. 3, pp. 536–539, 1969.

[13] R. Johnson, "Shallow water waves on a viscous fluid—the undular bore," *The Physics of Fluids*, vol. 15, no. 10, pp. 1693–1699, 1972.

[14] A. Saha, B. Talukdar, U. Das, and S. Chatterjee, "Solitons, compactons and undular bores in benjamin–bona–mahony-like systems," *Pramana*, vol. 88, no. 2, p. 28, 2017.

[15] A. Nayfeh, "Perturbation methods. a wiley-interscience publication," 1973.

[16] T. Shah, R. Chattopadhyay, K. Vaidya, and S. Chakraborty, "Conservative perturbation theory for nonconservative systems," *Physical Review E*, vol. 92, no. 6, p. 062927, 2015.

[17] A. H. Nayfeh and D. T. Mook, *Nonlinear oscillations*. John Wiley & Sons, 1979.

[18] V. Agrwal and H. Denman, "Weighted linearization technique for period approximation in large amplitude non-linear oscillations," *J. Sound Vib.*, vol. 99, no. 4, p. 463, 1985.

[19] Y. Cheung, S. Chen, and S. Lau, "A modified lindstedt-poincaré method for certain strongly non-linear oscillators," *Int. J. Nonlin. Mech*, vol. 26, no. 3-4, p. 367, 1991.

[20] Z. Feng and Y. Huang, "Approximate solution of the burgers-korteweg-de vries equation," *Communications on Pure & Applied Analysis*, vol. 6, no. 2, p. 429, 2007.

[21] S.-J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*. PhD thesis, Ph. D. Thesis, Shanghai Jiao Tong University Shanghai, 1992.

[22] J.-H. He, "A coupling method of a homotopy technique and a perturbation technique for non-linear problems," *Int. J. Nonlin. Mech.*, vol. 35, no. 1, p. 37, 2000.

[23] A. Yıldırım, "Homotopy perturbation method to obtain exact special solutions with solitary patterns for boussinesq-like b (m, n) equations with fully nonlinear dispersion," *J. Math. Phys.*, vol. 50, no. 2, p. 023510, 2009.

[24] J. Biazar and M. Eslami, "A new homotopy perturbation method for solving systems of partial differential equations," *Comput. Math. Appl.*, vol. 62, no. 1, p. 225, 2011.

[25] L. Cveticanin, M. Kalami-Yazdi, Z. Saadatnia, and H. Askari, "Application of hamiltonian approach to the generalized nonlinear oscillator with fractional power," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 11, no. 12, pp. 997–1002, 2010.

[26] P. Bera and T. Sil, "Homotopy perturbation method in quantum mechanical problems," *Appl. Math. Comput.*, vol. 219, no. 6, p. 3272, 2012.

[27] T. A. Nofal, G. M. Ismail, A. A. M. Mady, and S. Abdel-Khalek, "Analytical and approximate solutions to the fee vibration of strongly nonlinear oscillators," *J. Electromagn. Anal. Appl.*, vol. 5, no. 10, p. 388, 2013.

[28] K. Manimegalai, S. Zephania C F, P. Bera, P. Bera, S. Das, and T. Sil, "Study of strongly nonlinear oscillators using the aboodh transform and the homotopy perturbation method," *European Physical Journal Plus*, vol. 34, no. 1, p. 462, 2019.

[29] I. Mehdipour, D. Ganji, and M. Mozaffari, "Application of the energy balance method to nonlinear vibrating equations," *Current Applied Physics*, vol. 10, no. 1, pp. 104–112, 2010.

[30] M. Akbarzade and D. D. Ganji, "Coupled method of homotopy perturbation method and variational approach for solution to nonlinear cubic-quintic duffing oscillator," *Adv. Theor. Appl. Mech.*, vol. 3, p. 329, 2010.

[31] M. Madani, M. Fathizadeh, Y. Khan, and A. Yildirim, "On the coupling of the homotopy perturbation method and laplace transformation," *Math. Comp. Modelling*, vol. 53, p. 1937, 2011.

[32] V. Marinca and N. Herisanu, "Optimal homotopy perturbation method for strongly nonlinear differential equations," *Nonlinear Science Letters A*, vol. 1, no. 3, pp. 273–280, 2010.

[33] M. Hermann, M. Saravi, and H. E. Khah, "Analytical study of nonlinear oscillatory systems using the hamiltonian approach technique," *Journal of Theoretical and Applied Physics*, vol. 8, no. 3, p. 133, 2014.

[34] B. Wu, W. Liu, H. Zhong, and C. W. Lim, "A modified newton–harmonic balance approach to strongly odd nonlinear oscillators," *J. Vib. Eng. Technol.*, pp. https://doi.org/10.1007/s42417–019–00176–37, 2019.

[35] H. Ahmad, A. R. Seadawy, and T. A. Khan, "Numerical solution of korteweg–de vries-burgers equation by the modified variational iteration algorithm-ii arising in shallow water waves," *Physica Scripta*, vol. 95, no. 4, p. 045210, 2020.

[36] R. Hirota, *The direct method in soliton theory*. No. 155, Cambridge University Press, 2004.

[37] H. Hu, "Exact solution of a quadratic nonlinear oscillator," *J. Sound Vib.*, vol. 295, no. 9-10, p. 450, 2006.

6 Proposed Contents of the Thesis

The outline of the thesis is as follows:

1. Chapter 1 Introduction and Preliminaries
   (a) Section 1.1 Introduction
   (b) Section 1.2 Motivation
   (c) Section 1.3 Scope and Objective
   (d) Section 1.4 Outline of the thesis
2. Chapter 2 Autonomous conservative oscillator
   (a) Section 2.1 Introduction
   (b) Section 2.2 Formalism of LH
   (c) Section 2.3 Results and discussions
   (d) Section 2.4 Conclusion
3. Chapter 3 Korteweg-de Vries- Burgers equation
   (a) Section 3.1 Introduction
   (b) Section 3.2 Results and discussions
   (c) Section 3.3 Conclusion
4. Chapter 4 Kadomtsev-Petviashvili equation
   (a) Section 4.1 Introduction
   (b) Section 4.2 Results and discussions
   (c) Section 4.3 Conclusion
5. Chapter 5 Symmetric and asymmetric oscillators with polynomial restoring forces
   (a) Section 5.1 Introduction
   (b) Section 5.2 Formalism of LHh
   (c) Section 5.3 Results and discussions
   (d) Section 5.4 Conclusion
6. Chapter 6 Conclusion and Future work

7 Publications based on the research work

7.1 Journal Publications

1. **C. F. S. Zephania**, T. Sil "*A Generalized Accurate Approximate Solution to the Symmetric and Asymmetric Oscillators with Polynomial Restoring Forces* ",Journal of Vibration Engineering & Technologies, 9, 1059, 2021, doi: 10.1007/s42417-021-00282-1.
2. **C. F. S. Zephania**, T.Sil "*Study of autonomous conservative oscillator using an improved perturbation method* ",Journal of Vibration Engineering & Technologies ,9, 3, 2021, doi: 10.1007/s42417-020-00233-2.
3. K. Manimegalai, **C. F. S. Zephania**, P. K. Bera, P. Bera, S. K. Das, T. Sil "*Study of strongly nonlinear oscillators using the Aboodh transform and the homotopy perturbation method* ", European Physical Juounal Plus, 34, 1, 2019, doi:10.1140/epjp/i2019-12824-6.
4. **C. F. S. Zephania**, P. C. Harisankar, T. Sil "*An improved perturbation method to study Korteweg-de Vries-Burgers equation* (under review)

7.2 Conferences Presentations

1. **C. F. S. Zephania**, T. Sil., "*An Improved Homotopy Perturbation Method to Study Damped Oscillators.*, Accepted in (ICTACEM 2021)
2. **C. F. S. Zephania**, P. C. Harisankar, T. Sil. "*Solution of the Kadomtsev-Petviashvili equation using an improved homotopy perturbation method* in ICAPSM Conference Proceedings (ICAPSM 2021)
3. **C. F. S. Zephania**, P C Harisankar, T. Sil., "*Improved homotopy perturbation method to solve Duffing oscillator*, in CNSD Conference Proceedings (CNSD 2019)